Packing problems of objects in science and in everyday life

Docent lecture by Dr. Martin Trulsson

27/9-2019

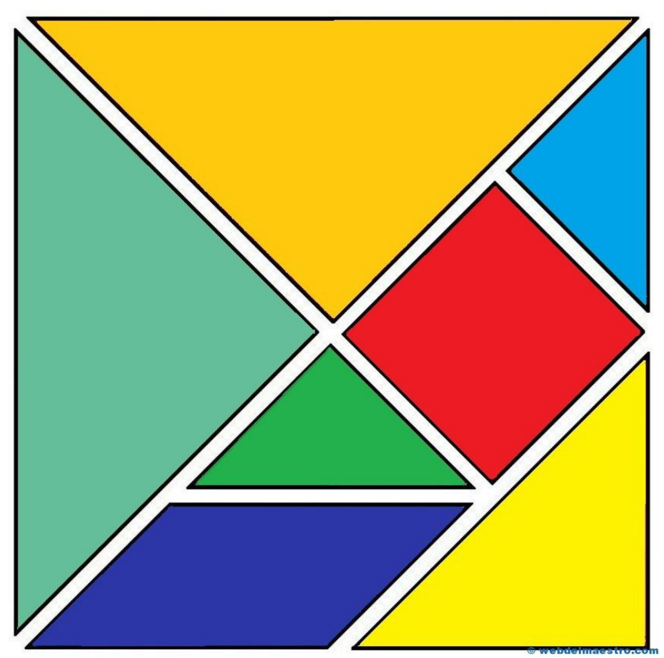
Geometric packing problems (i.e. not considering weight and etc)

- Optimisation problem

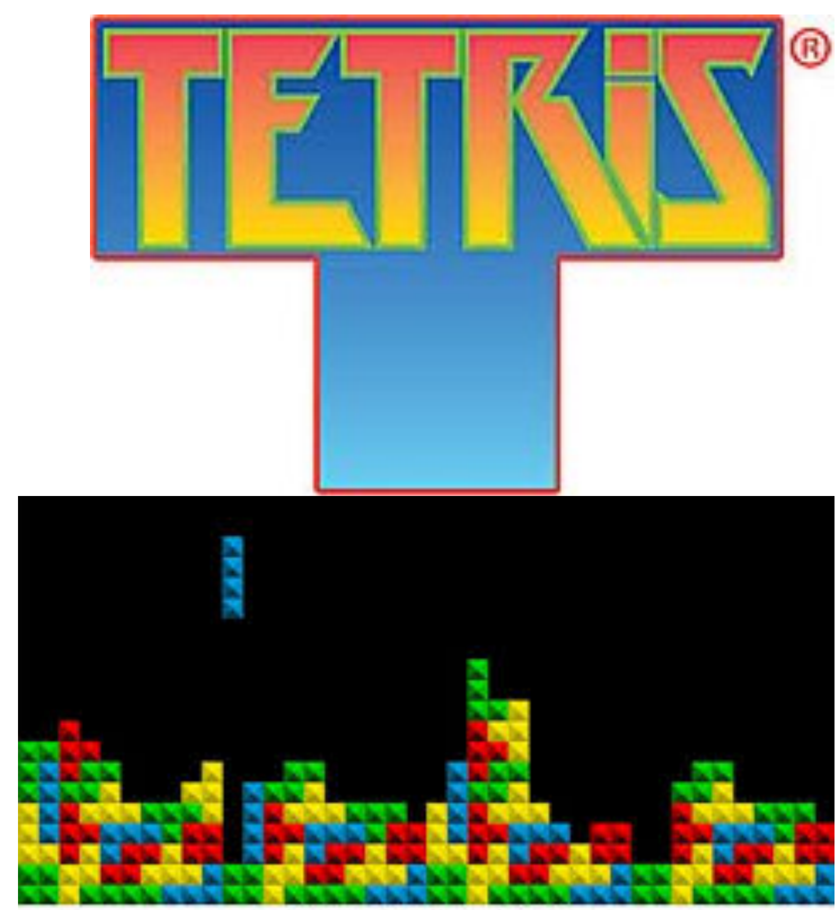
Important for packaging (objects: pills, milk carton, cloths etc), storage and/or transport (in flights)

Bin packing problem:

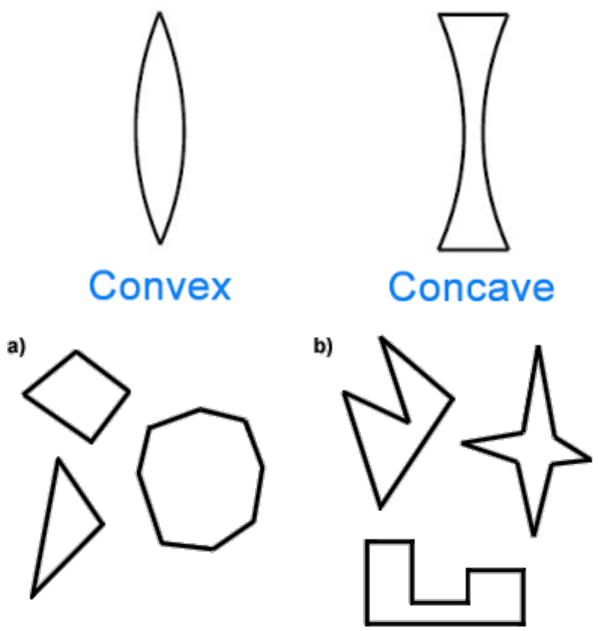
1) Having a container
2) Pack a set (maximum) of objects into the container



Tangram

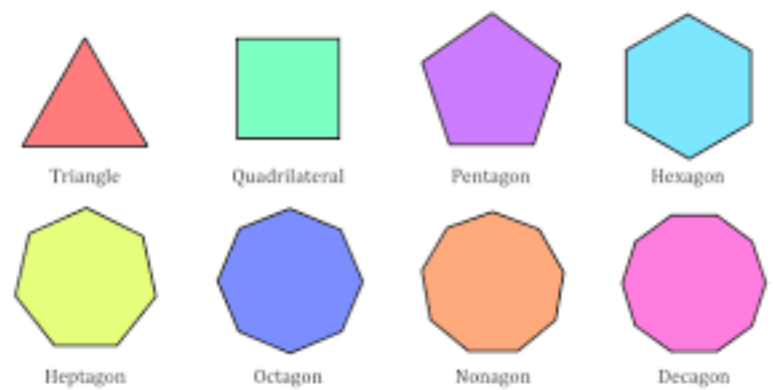


Infinite space (or periodic packing)
Avoiding the voids (and overlaps)



Regular (convex) polygons

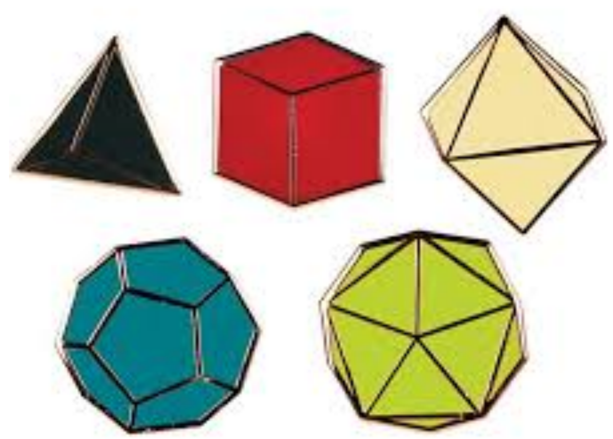
2D



Infinite number

Regular (convex) polyhedrons

3D

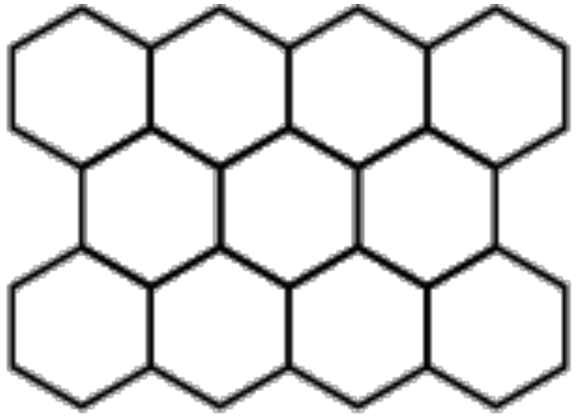


Platonic solids

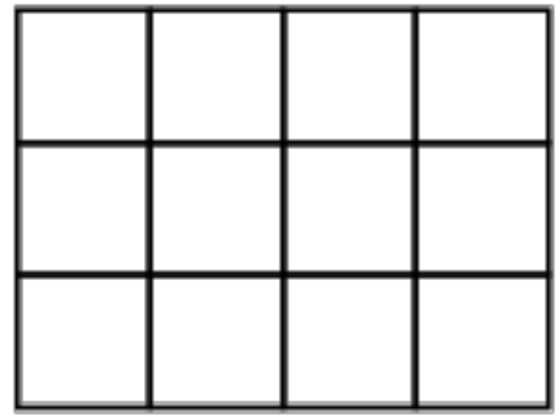
5 of them

Avoiding the voids

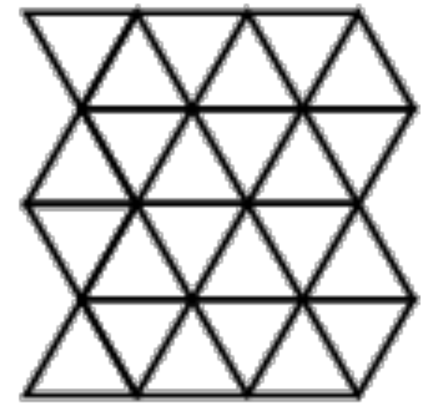
In two-dimensions there exits three regular tiling that are area-filling
(Out of the infinite regular polygons)



Hexagon



Square



Triangle

Settlers of Catan

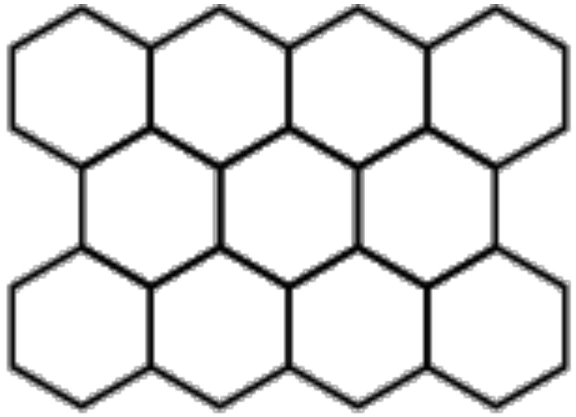
Chess

Gets

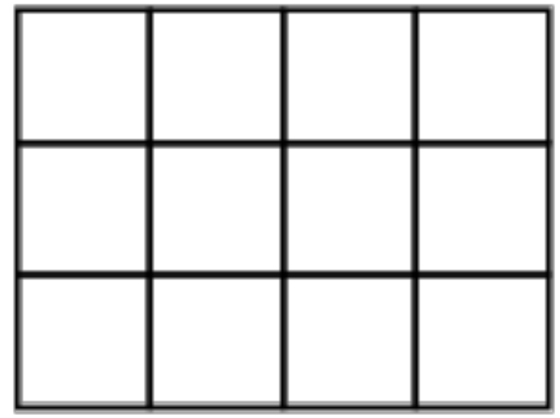
Board games (also frequently used in video games)

Avoiding the voids

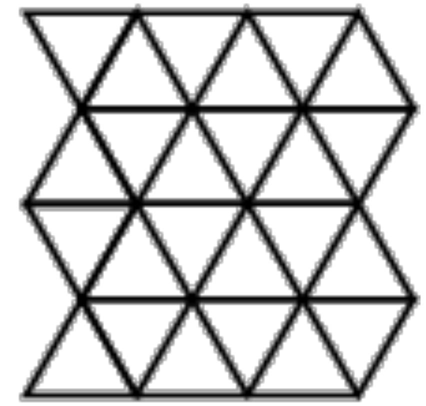
In two-dimensions there exits three regular tiling that are area-filling
(Out of the infinite regular polygons)



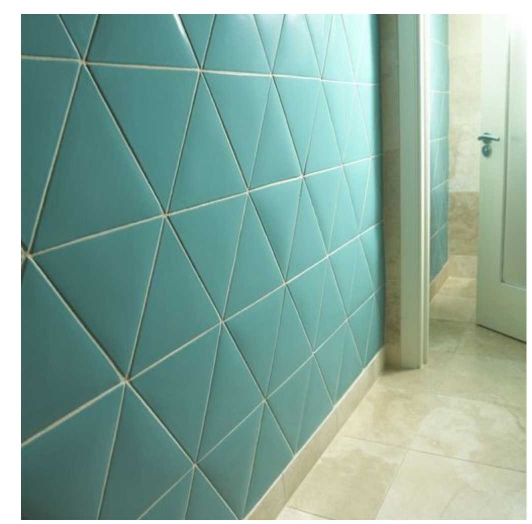
Hexagon



Square



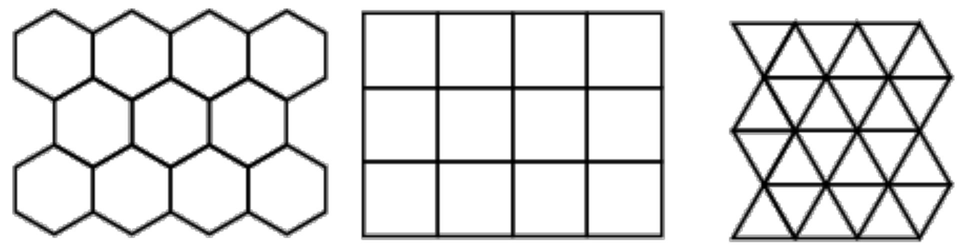
Triangle



Or just some inspiration to your kitchen or bathroom!

Tessellations

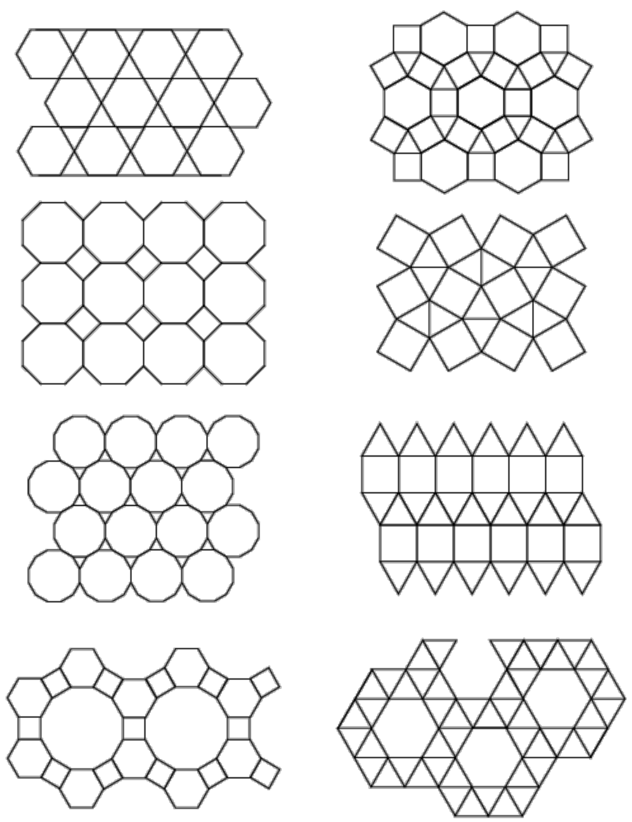
These periodic tilings without overlap
or voids are called tessellation's



There exist three **regular tessellations** in 2D (above)

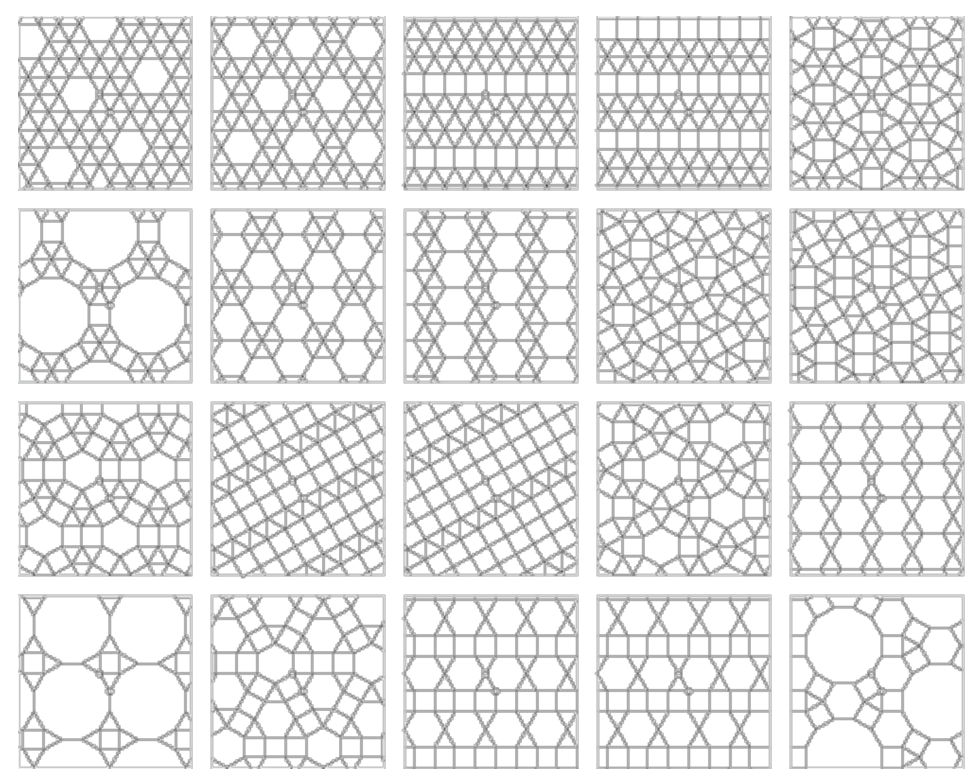
Semiregular (Archimedean) tessellations

Two or three polygons (share vertexes)



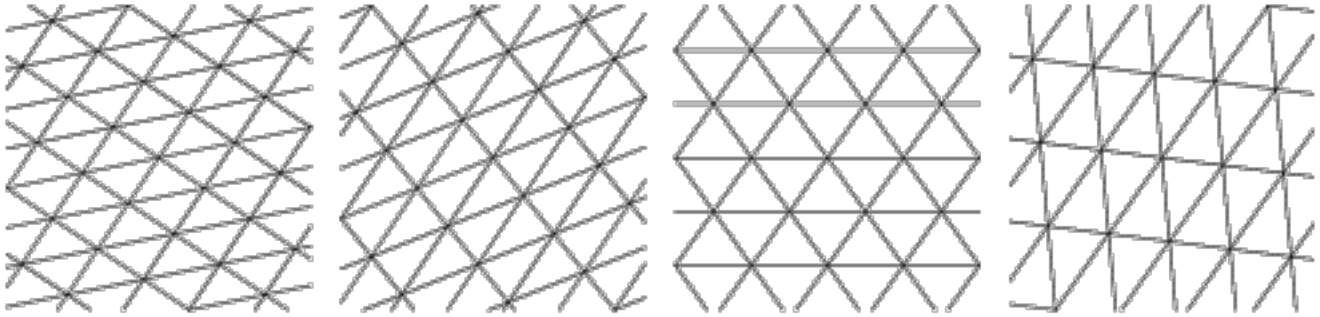
Demiregular tessellations

Two or three polygons (not share vertexes)

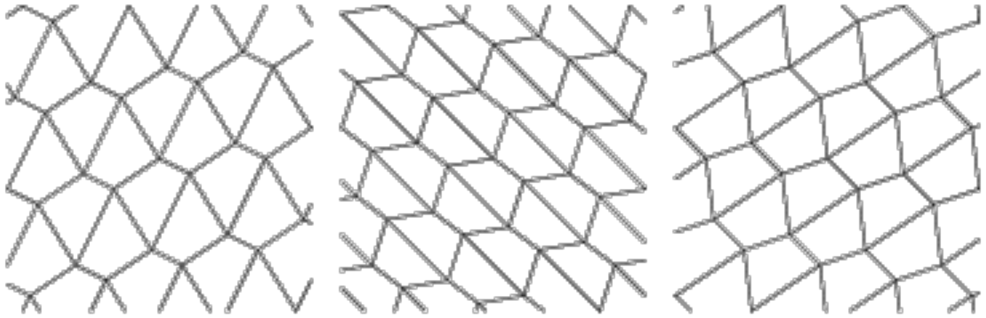


Other tilings (of non-regular but convex objects)

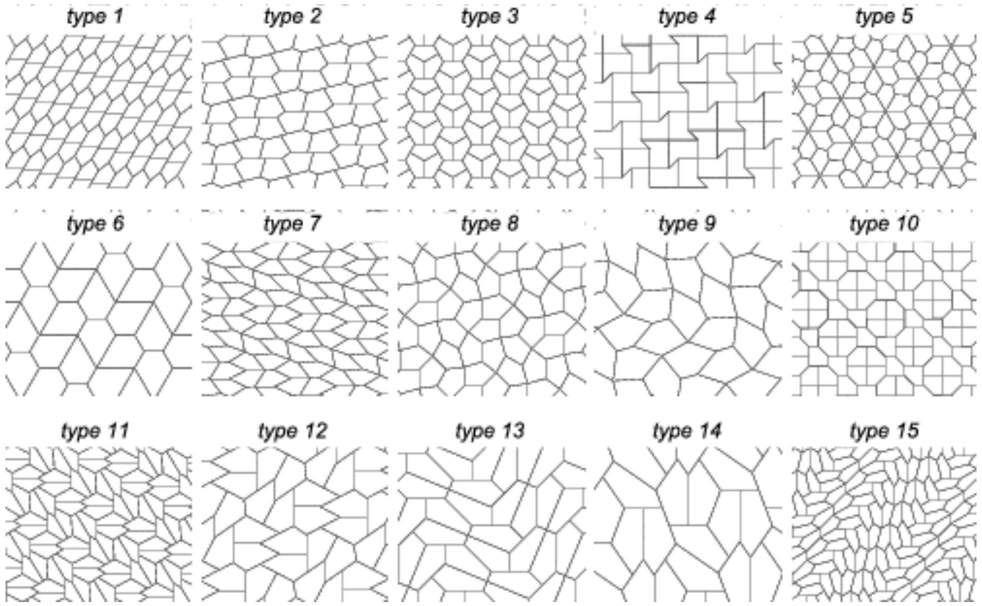
Triangle



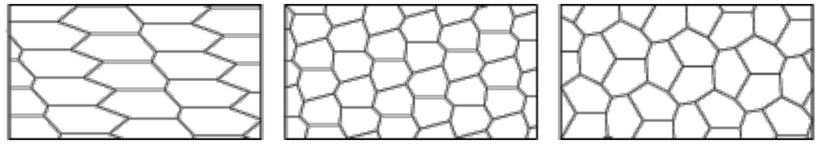
Quadrilateral (four sides)



Pentagon



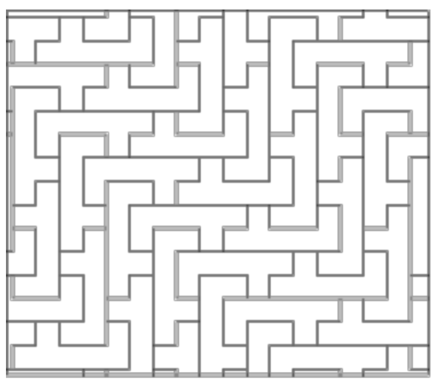
Hexagon

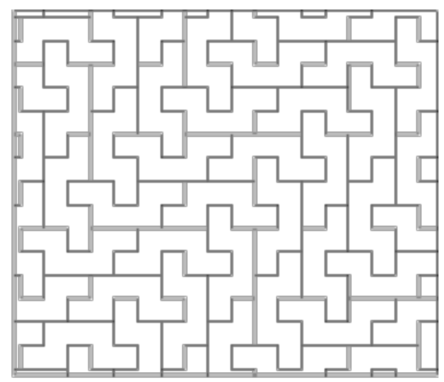


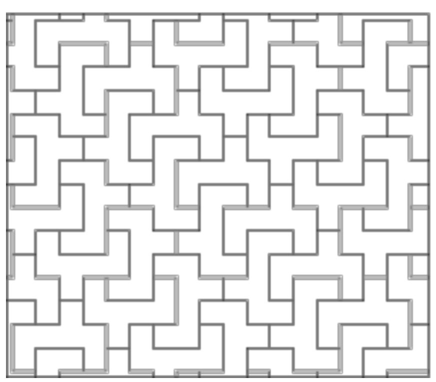
sides>6 only non-convex

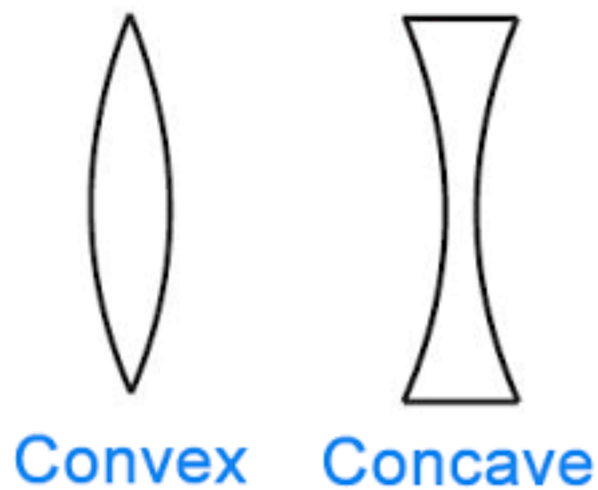
Many other area-filling tilings with non-convex or not edge-matching polygons

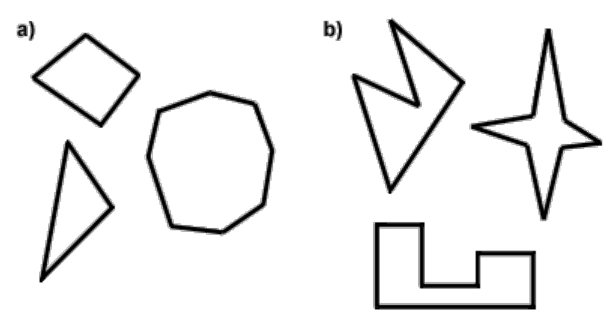
Non-convex

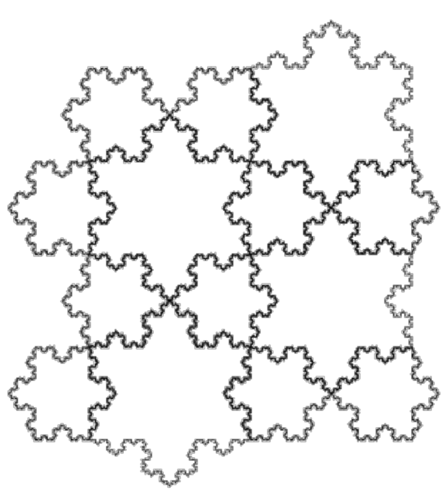






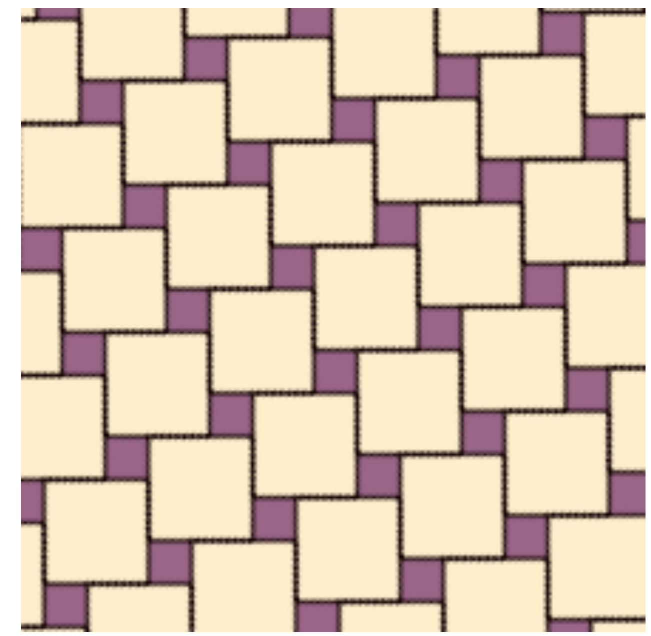


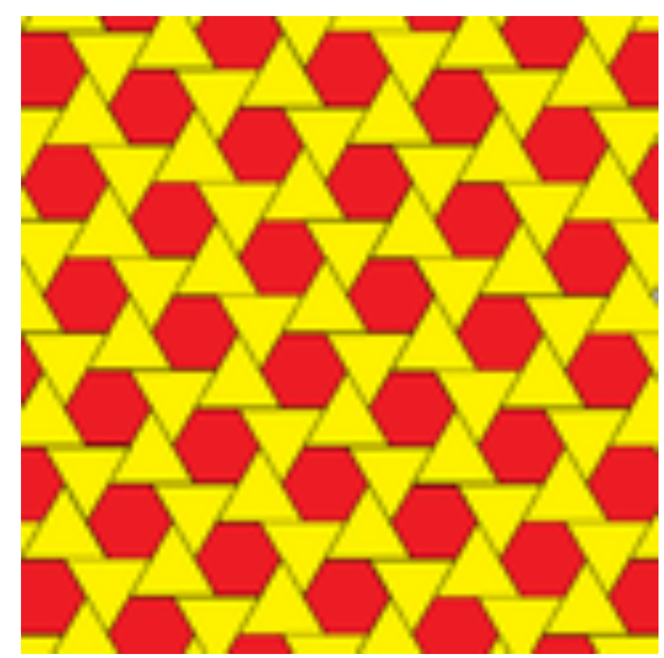




Star tilings

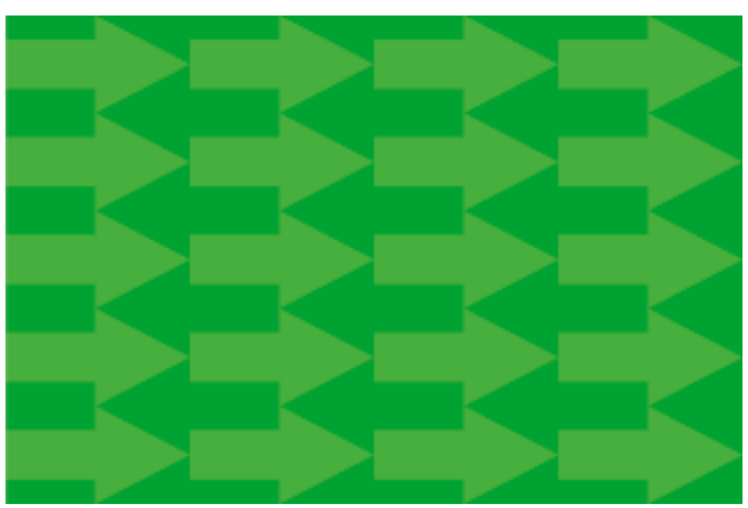
Convex but not edge-to-edge





More periodic tilings

Archelogical museum of Seville, Spain



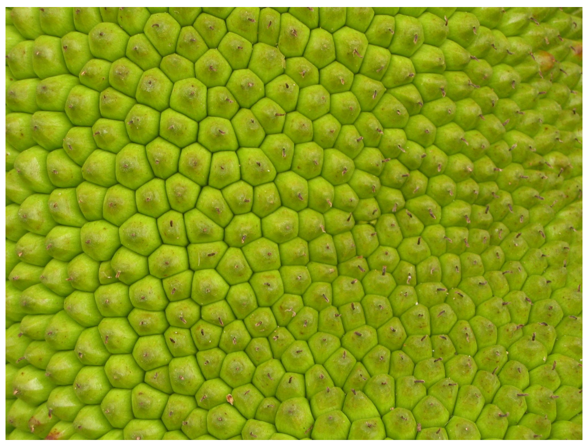
Skånetrafiken

Cairo street tiling

Two birds, M.C. Esher

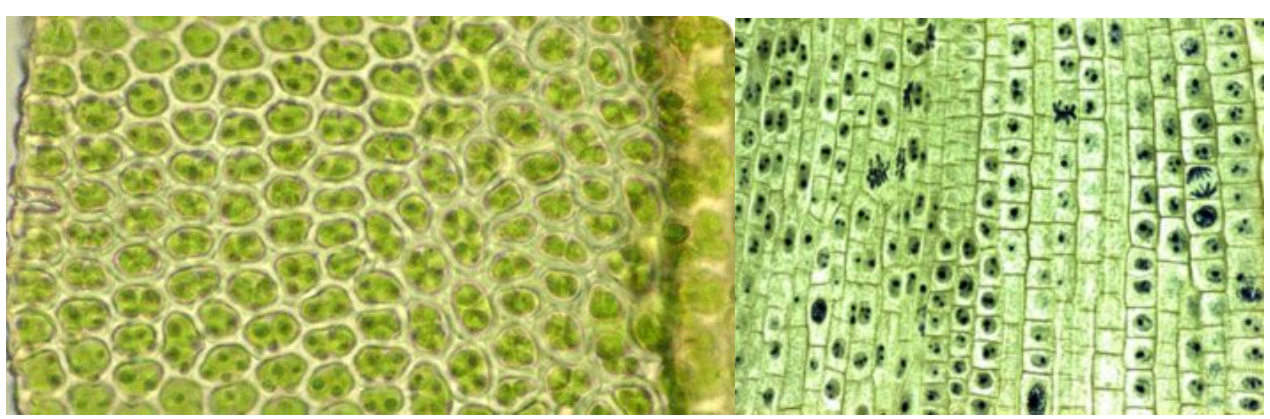
Tomb of Moulay Ishmail,
Meknes, Marrocco

Tillings in nature



Jackfruit texture

Honeycomb of bees



Hexagonal and Cubical Plant Cells

Tillings in nature continued (curvature)

Snake scales

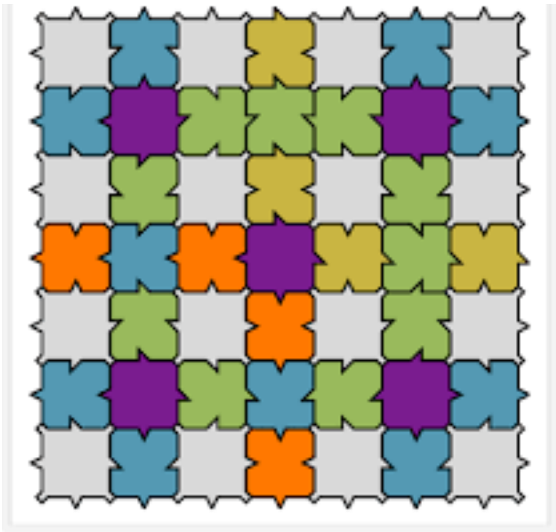
Pineapple

A fly's eye

Cone of spruce

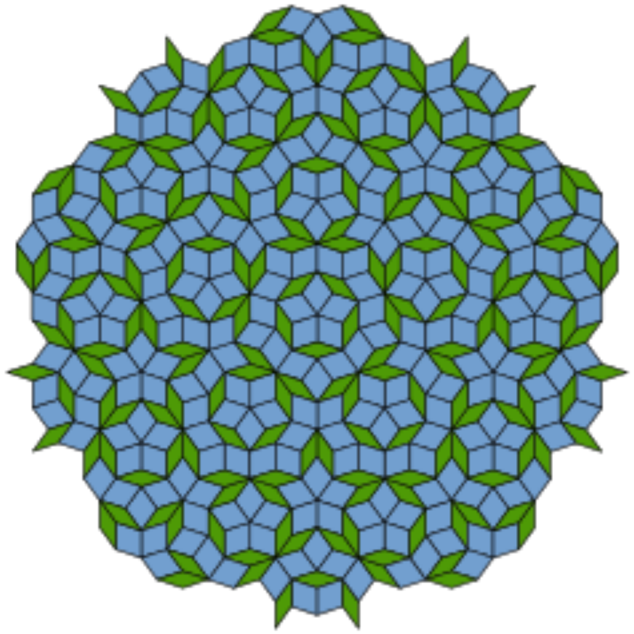
Aperiodic Tilings

Lacks translational symmetry
but is self-similar



Robinson tiling (6 building blocks)

1971

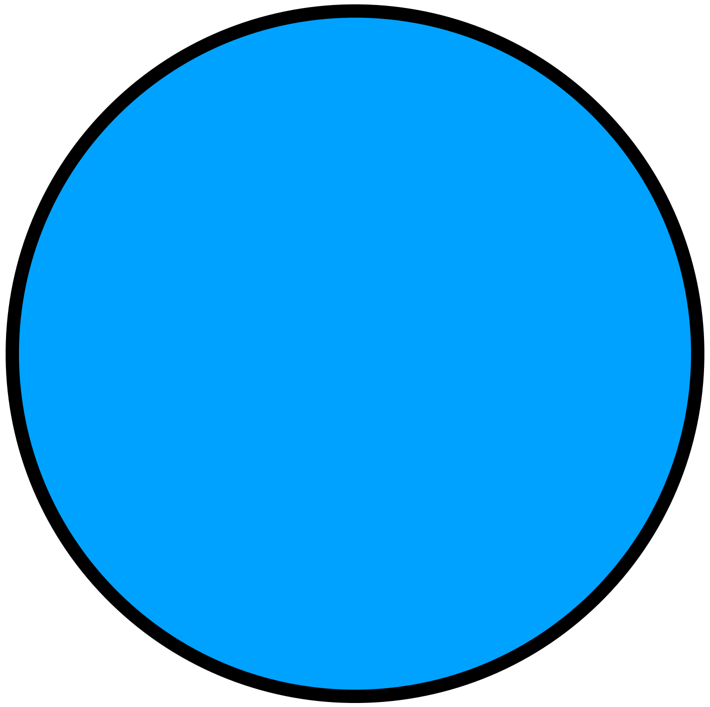


Penrose tiling (2 building blocks)

1974

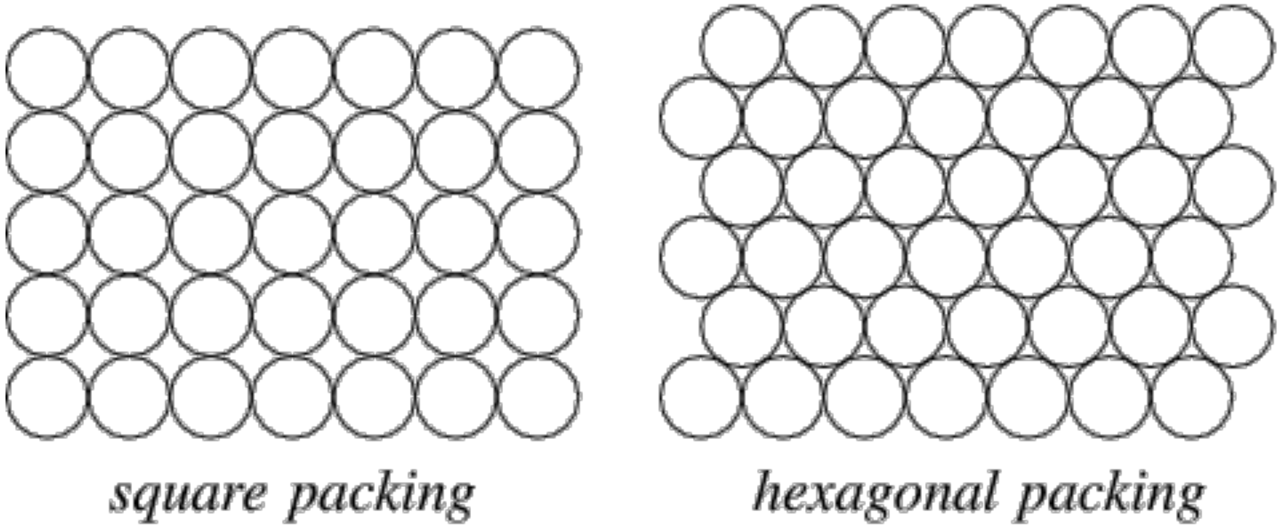
Open question:
Aperiodic tilings with only one
"connected" building block (Einstein)?

Other simple objects



The disc

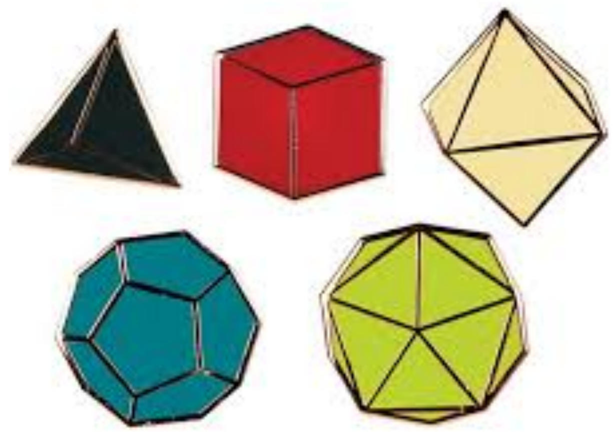
Disc packings



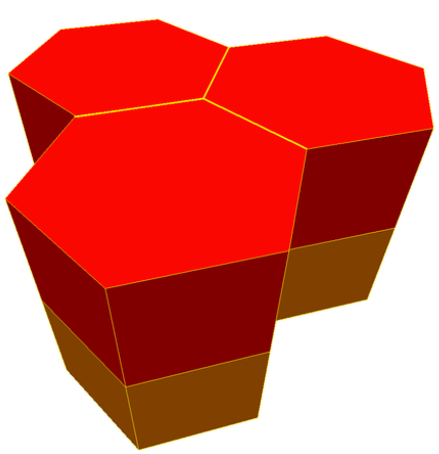
$$\phi = \frac{\pi}{4} \approx 0.7853981634 \qquad \phi = \frac{\pi\sqrt{3}}{6} \approx 0.9068996821$$

That the hexagonal is the densest was proved by Gauss (1831, assuming regular lattices)
Fejes Tóth (1940) for all possible plan packings

Avoiding the voids in 3D
(or tessellations in 3D)



Among the platonic solids only the cube turns out to be space-filling (i.e. no voids)



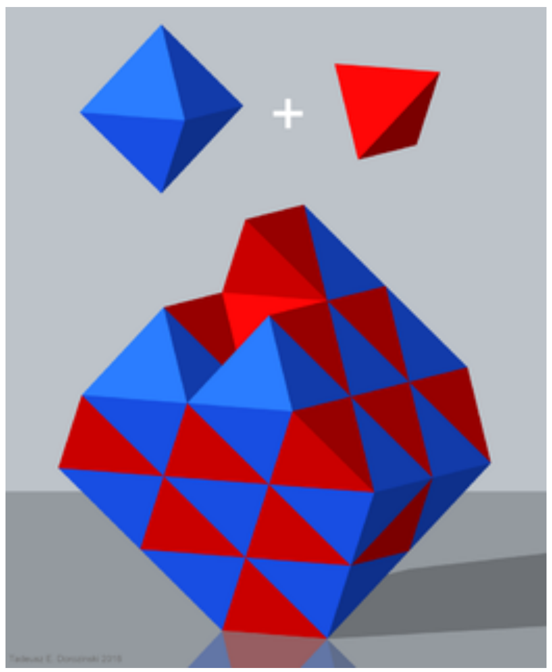
Hexagonal prism

Rhombic dodecahedron

Truncated octahedron

Use two or more sizes:

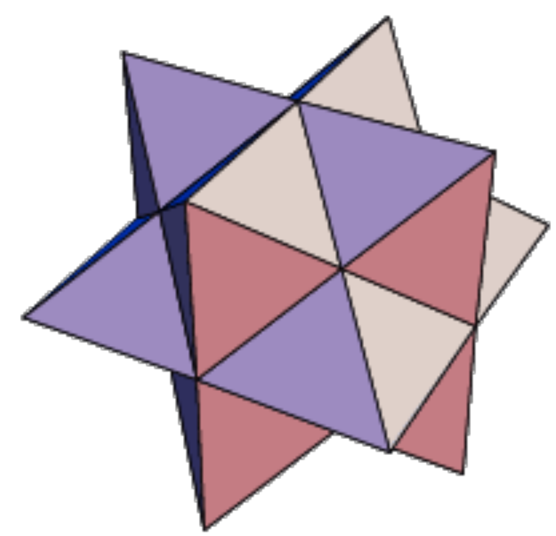
Combinations of platonic solids may however fill the space



Regular octahedra and tetrahedra in a ratio of 1:2

Or more complex objects:

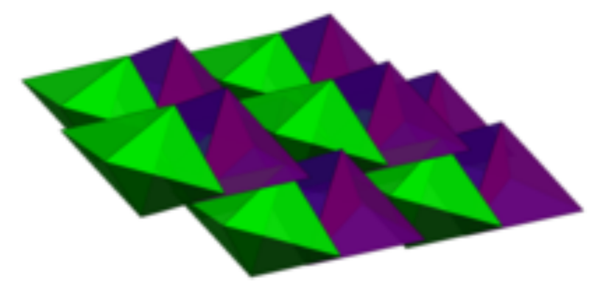
Esher’s solid



But wait? Can I not pack the space with tetrahedrons?

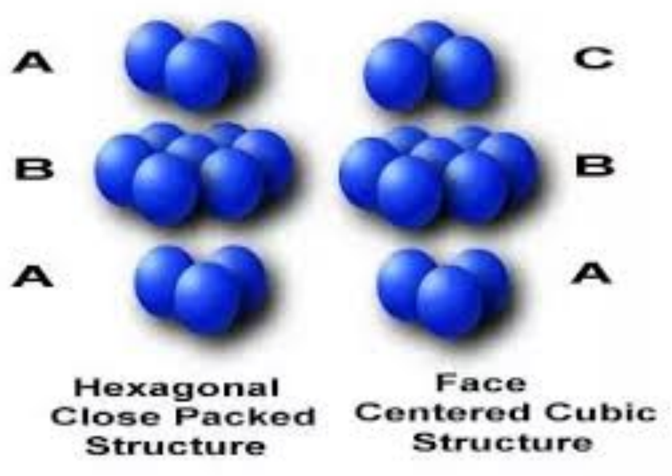
Claimed by Aristotle!

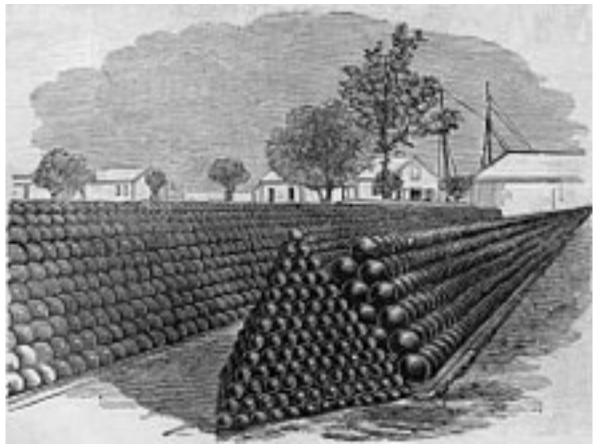
Not the case!
But then: What is then the densest of tetrahedrons?



- **1970** Hoylman proved that the densest lattice packing was 36.73% (“Bravais lattice”)
- **2006** Conway and Torquato found a non-Bravais lattice of tetrahedrons of rough 72%
- **2009** Chaikin showed experimentally that random dices of tetrahedrons pack denser 76%
- **2009** Glotzer showed by large-scale simulations that it could be even denser (via quasi-crystal) 85.03%
- Later that **same year** Gravel et al. showed by a simple unit cell that one could reach 85.47%
- “**Competition**” between different groups then continued and today densest is known to be 85.63%
- Maybe there exists a denser packing???

OK, what about spheres?





Two kind of packings (based on hexagonal layers)

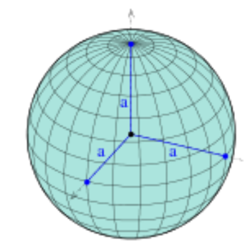
Hcp (hexagonal close packing): ABABAB....
Fcc (face centered cubic packing): ABCABC...

$$\phi \approx 74\%$$

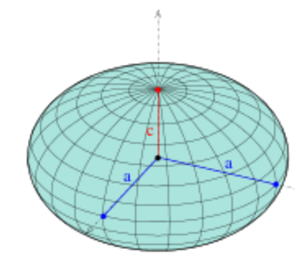
Is this the densest packing of spheres?
Kepler says yes

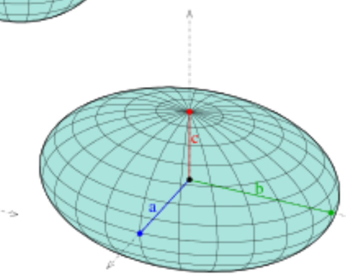
First real (mathematical) proof by Hales 1998 (including non-periodic lattices etc). But reviewers were certain to 99%. 2014 (almost 20 years) Hales used computers to validate that he's proof was indeed correct.

OK, what about packing ellipsoids?

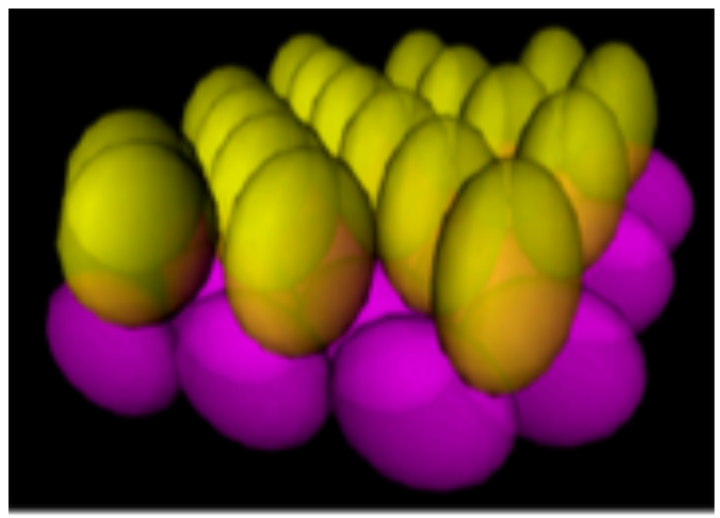


The densest know is (2004) $\phi \approx 77\%$



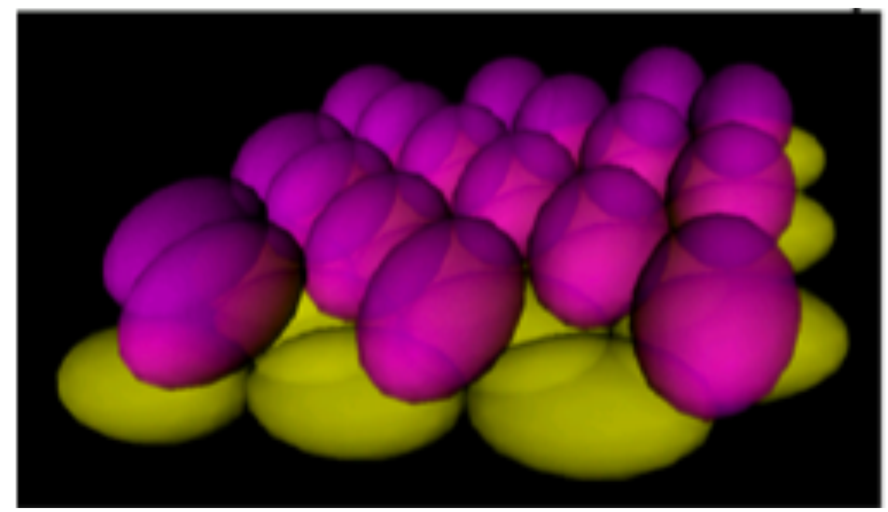


Denser than for spheres!



$\alpha < 1/\sqrt{3}$

Oblate



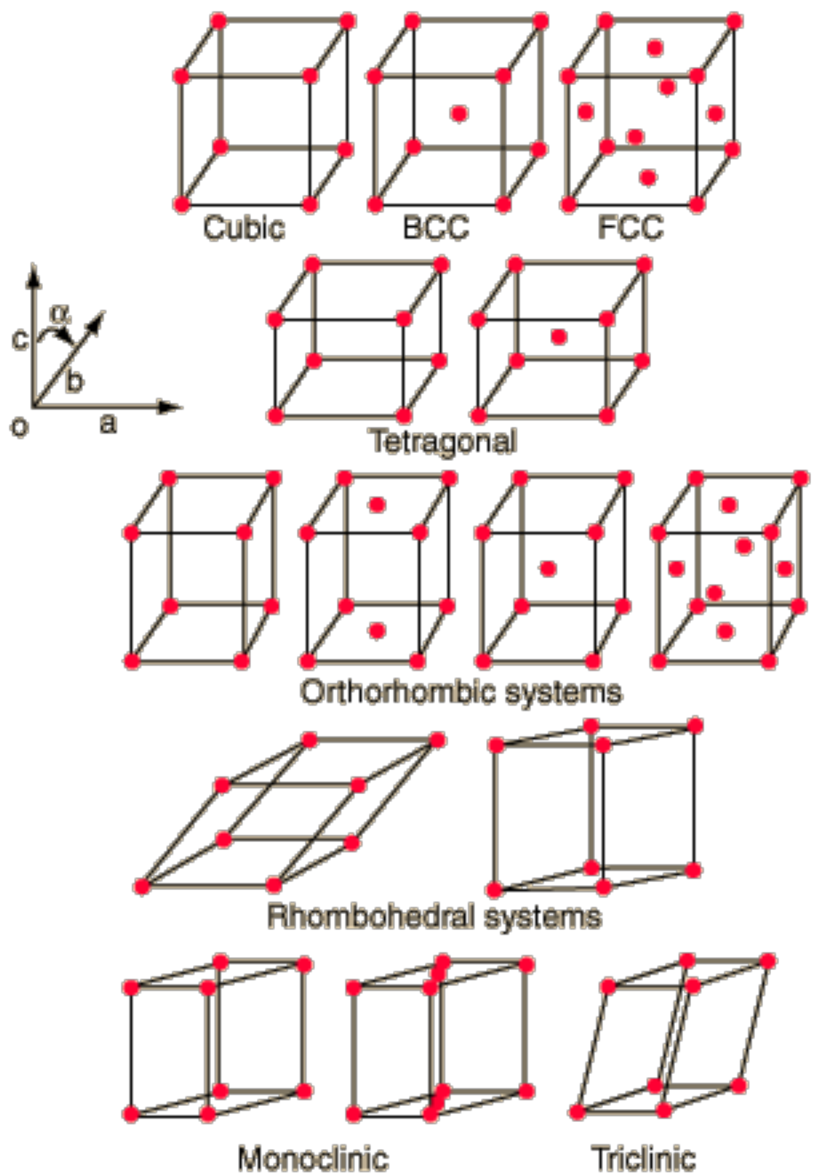
$\alpha > \sqrt{3}$

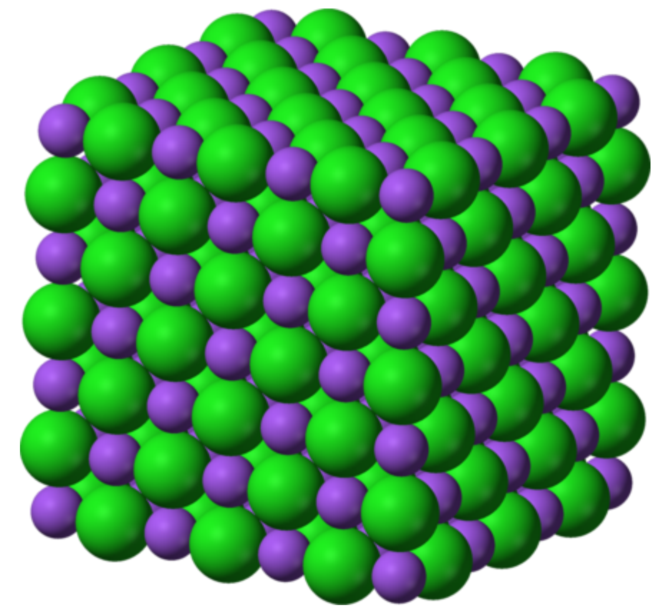
Prolate

Even denser ones??

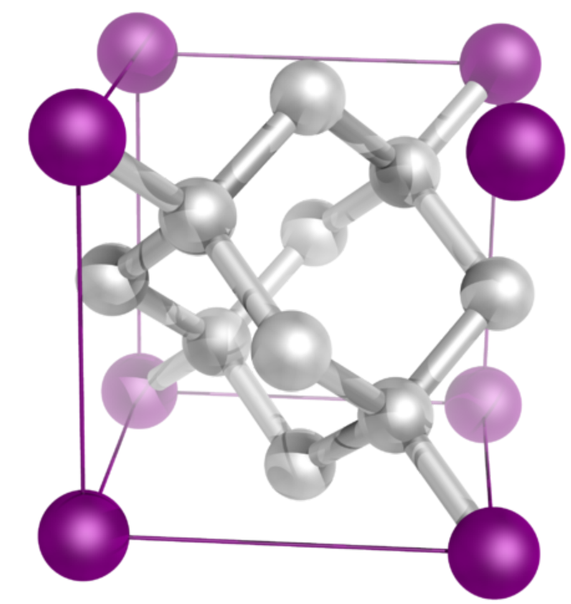
Crystal structures with atoms or molecules

Categories into 14 Bravais lattices
(or 7 crystal systems)





NaCl (fcc)



Diamond (fcc)

Crystallography!!
Number of Nobel prices related to this is huge
(including Röntgen, Bragg and Quasicrystals)

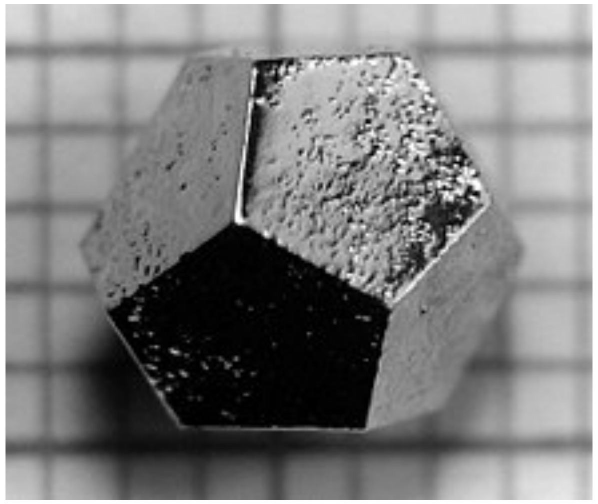
Material science:
mechanical, optical and other various properties

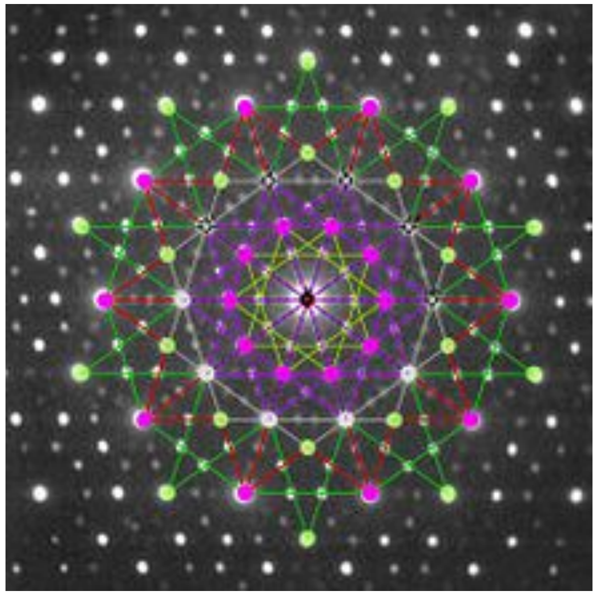
Aperiodic Tessellation in 3D

Conway, Penrose, Ammann (amateur)

"Mathematical puzzles"

1982 - Quasicrystals (Dan Shechtman, Nobel prize 2011)





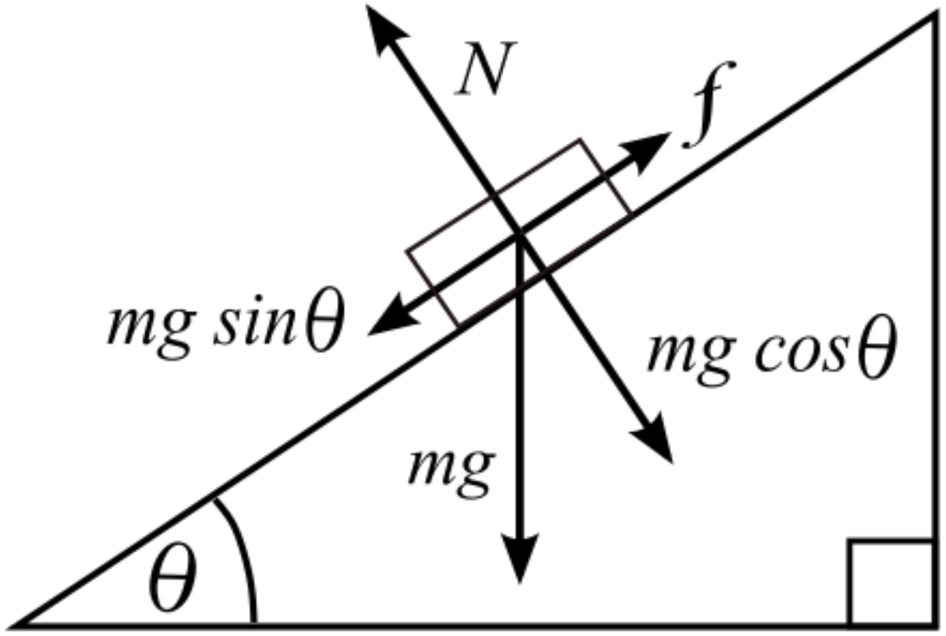
Ho-Mg-Zn dodecahedral quasicrystal

Linus Pauling: "There is no such thing as quasicrystals, only quasi-scientists"

Related topic: Pilling spheres

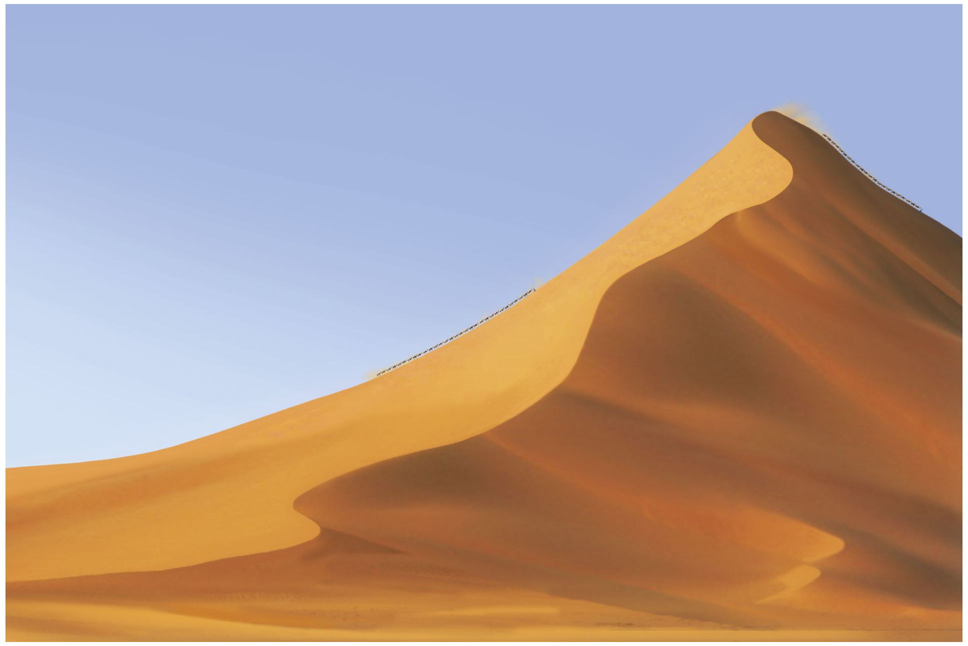
Need of friction at least on the base plan

Tennis ball towers (without glue)



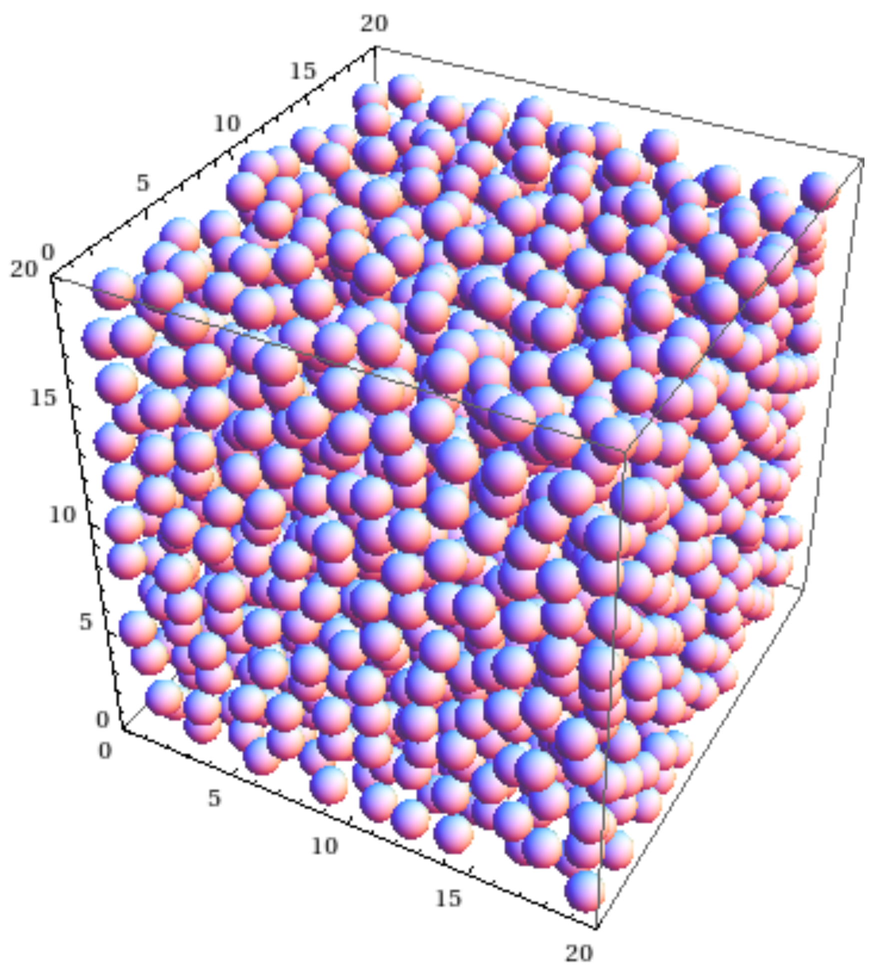
Sand and gravel (or stones)

Not as regular or with different shapes and sizes. How do they pack?



Random close packing (or jamming)

Assuming hard spheres with a slight
poly-dispersity (different sizes) or with a slight asymmetric shape



$$\phi \approx 64\%$$

Frictionless

How do I know it's a solid?

Short answer: It does not flowing (upon forcing)
I.e. need finite yield stress/force (mechanical stable)

Gravity

Not good $Z = 2$

Good, even if marginally $Z = 3$

Very good (too much)? $Z = 4$

For spheres the constrain is (3D):

$Z_{\text{iso}} = 2d$ (6) Dense (lubricated)

$Z_{\text{iso}} = d + 1$ (4) Loose (frictional)

Packing “frictional” spheres

Very loose random packing by letting spheres settle in a viscous fluid

$$\phi \approx 0.56$$

Loose random packing by dropping or packed by hand

$$\phi \approx 0.59 - 0.60$$

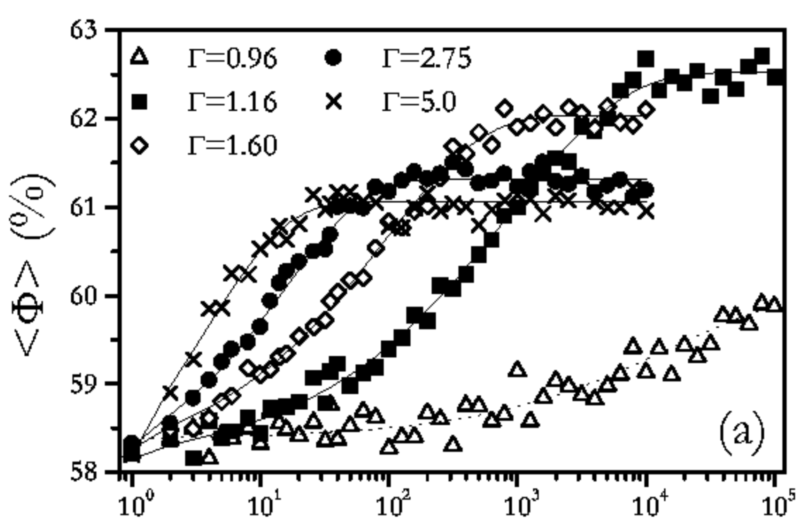
Poured random packing

$$\phi \approx 0.61 - 0.62$$

Close random packing (by vibrations or tapping the jar)

$$\phi \approx 0.62 - 0.64$$

$$\phi_{\mathrm{RCP}} \approx 0.64$$



Tapping

The trap

Jamming, friction, force chains and rattlers

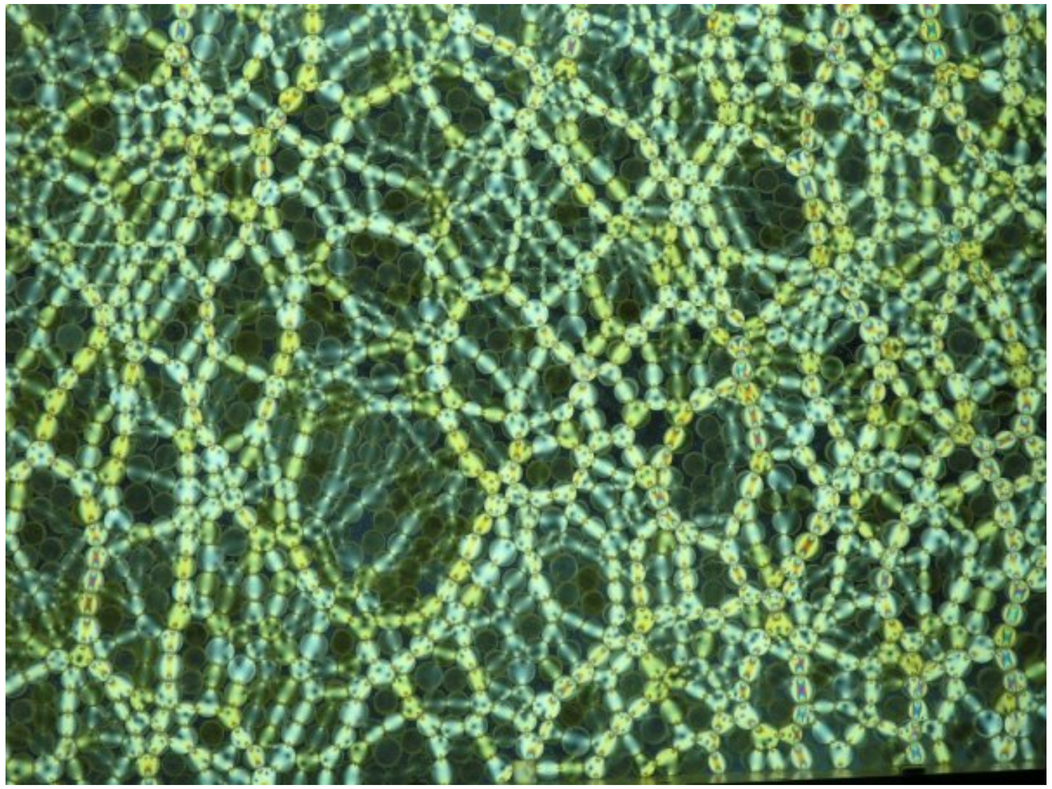


Photo-elastic discs

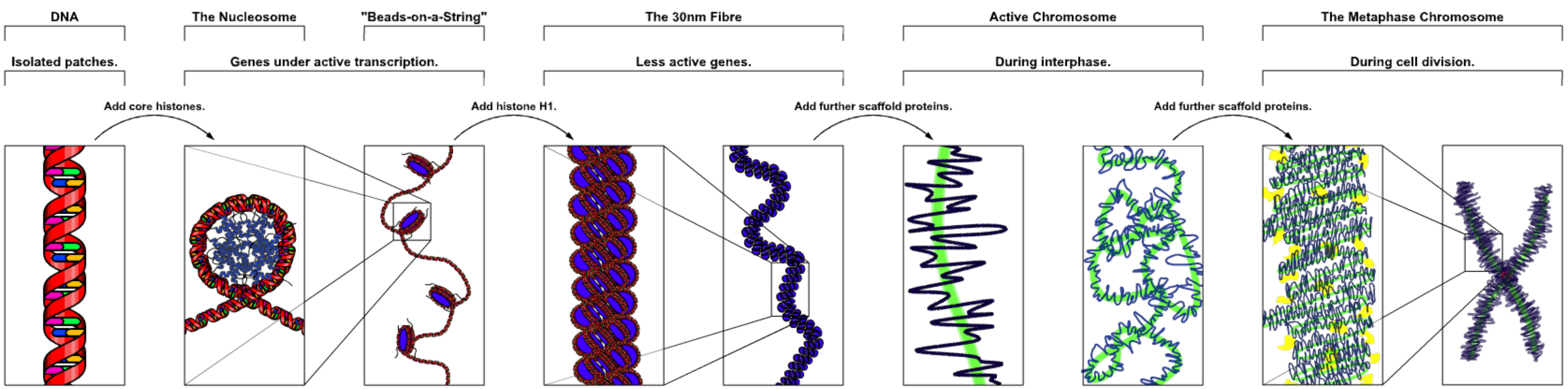
Random close packing II

Adding instead asymmetry as for M&M's

$\phi \approx 74\%$

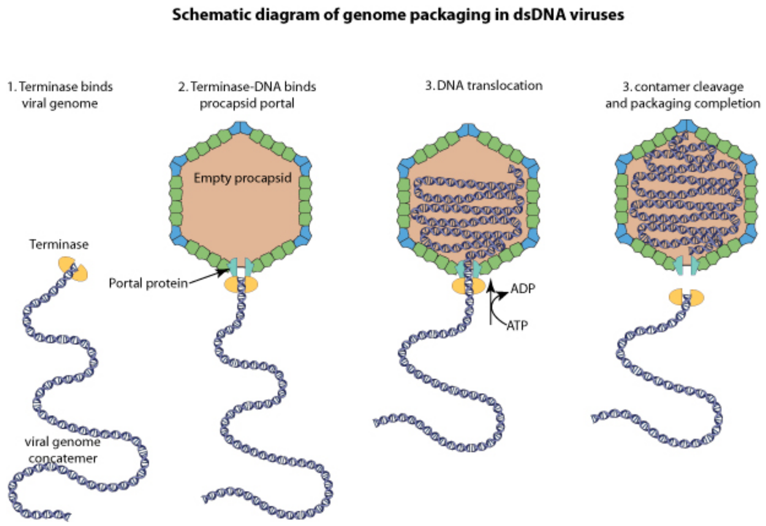
Much denser than for RCP spheres

Packing in biological systems

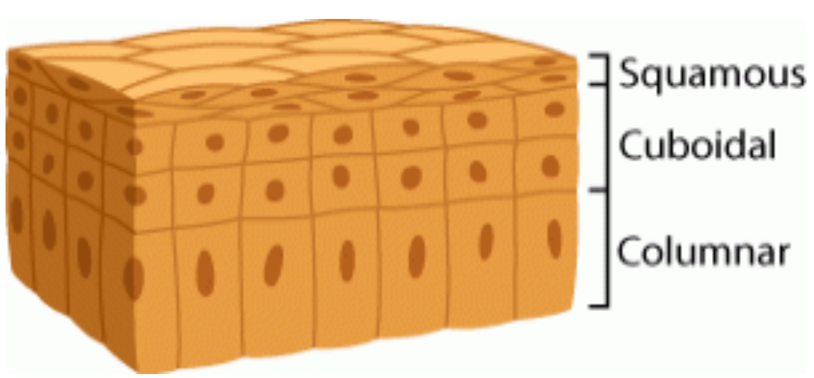


Hierarchical packing of DNA into chromosomes

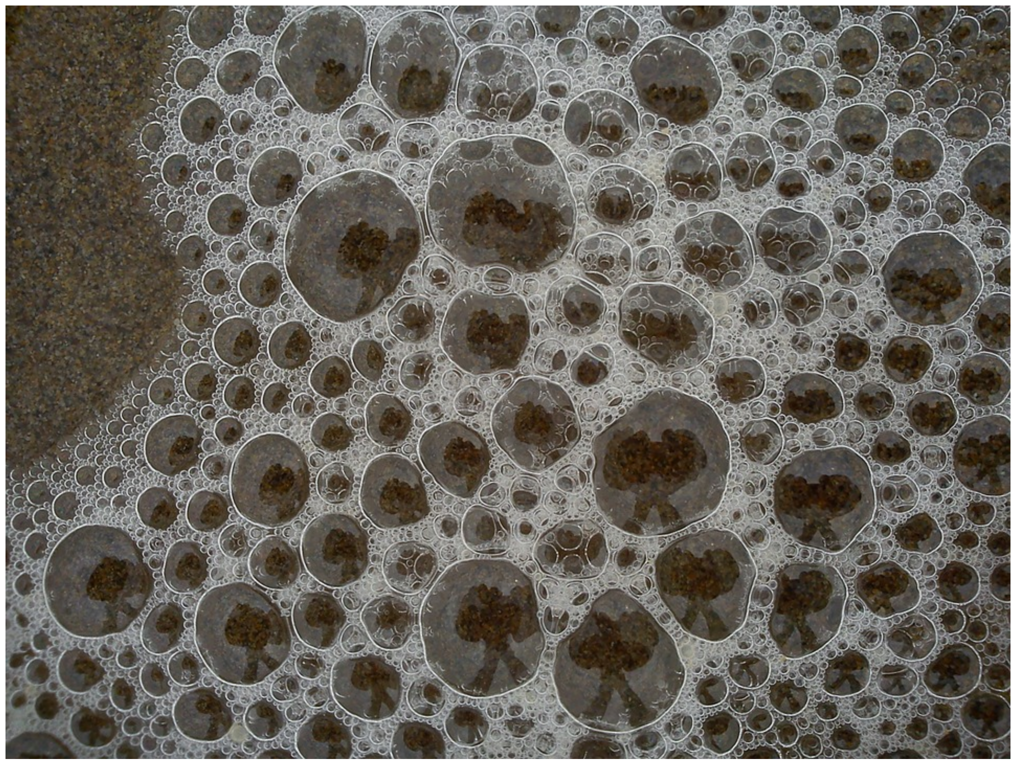
Packing genome in an efficient way

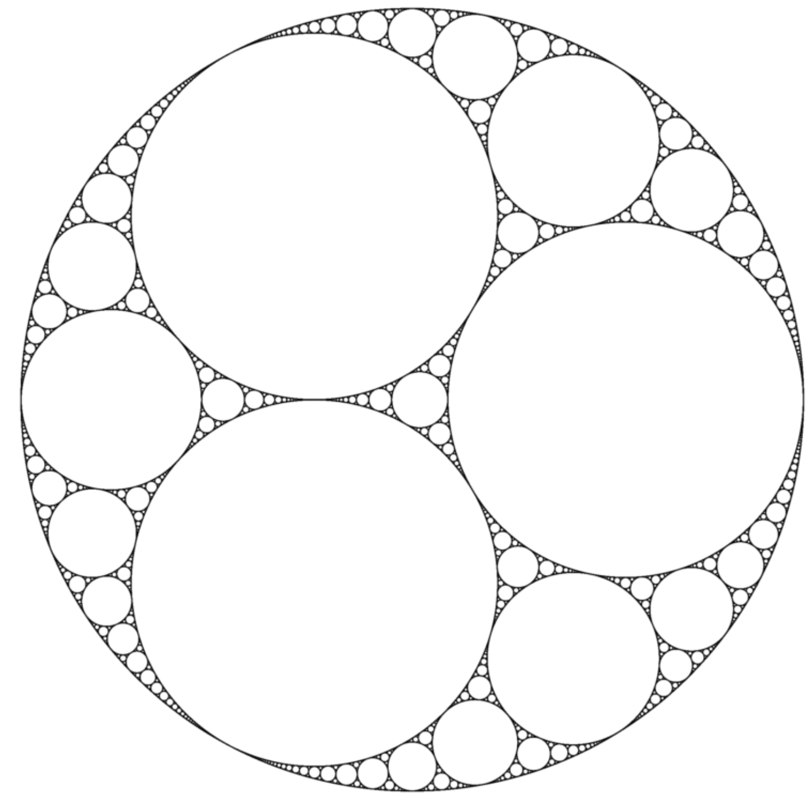


Cell packings



Packing of bubbles/foams



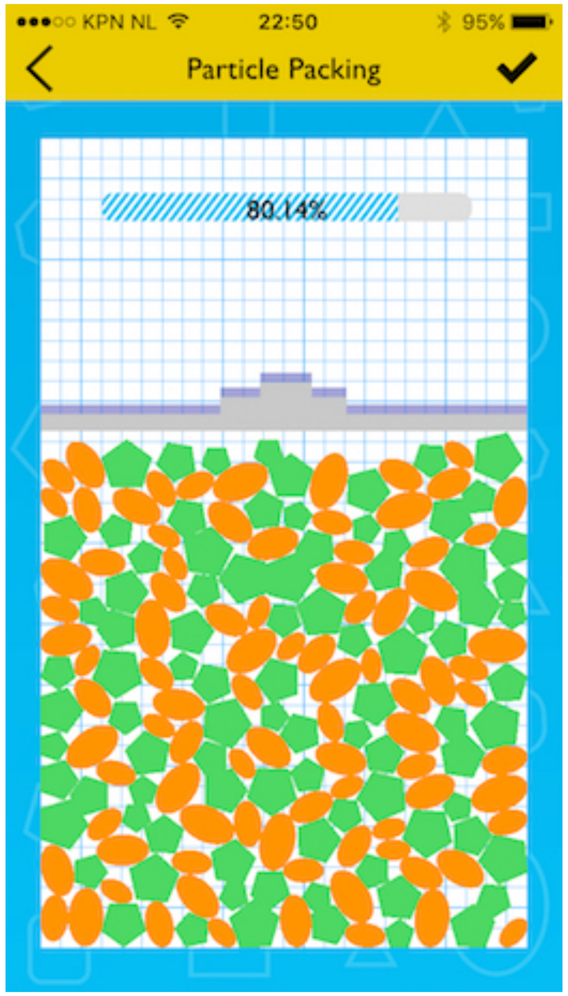


Apollonian gasket

1. Draw a large circle
2. Put 3 circles within and touching the large one
3. Fill the gaps with the biggest circles and that are touching the other circles
4. Continue with 2

Forms a so called fractal structure

Why not start yourselves?



Download the app “Geyopp”
and try to maximise the packing

Competition time!
How many “geléhallon” are there in the jar?

Thank you!

Competition time!
How many “geléhallon” are there in the jar?

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